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Path integral quantisation of finite noncommutative geometries

Mark Hale

Department of Mathematical Sciences, Science Laboratories, University of Durham, Durham DH1 3LE, UK

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Abstract

We present a path integral formalism for quantising gravity in the form of the spectral action. Our basic principle is to sum over all Dirac operators. The approach is demonstrated on two simple finite noncommutative geometries: the two-point space, and the matrix geometry $M_2(\mathbb{C})$. On the first, the graviton is described by a Higgs field, and on the second, it is described by a gauge field. We start with the partition function and calculate the propagator and Greens functions for the gravitons. The expectation values of distances are evaluated and we discover that distances shrink with increasing graviton excitations. We find that adding fermions reduces the effects of the gravitational field. A comparison is also made with Rovelli's canonical quantisation approach, and with his idea of spectral path integrals. We include a brief discussion on the quantisation of a Riemannian manifold. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the greatest successes of noncommutative geometry has been the unification of the forces of nature into a single gravitational action—the spectral action [2,3]. This has been achieved at the classical level for an Euclidean signature. It does this by using the Kaluza–Klein idea of rewriting all the gauge fields as components of a metric on a more

E-mail address: m.hale@qmul.ac.uk (M. Hale).

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structured spacetime. Noncommutative geometry succeeds where Kaluza–Klein fails as it is not limited to conventional differential manifolds. For introductions to noncommutative geometry see [4–6].

The question of how to quantise a field theory on a general noncommutative geometry remains largely unresolved. Conventional techniques work on Riemannian-like manifolds and have been used on noncommutative extensions, such as almost commutative geometries (the tensor product of a Riemannian manifold with a finite noncommutative geometry) and the noncommutative torus [7]. Beyond this, most efforts have focused on quantising a particular noncommutative geometry [8–10].

In this paper, we present a path integral approach that is applicable to any noncommutative geometry. It has been developed to quantise the spectral action, which is the natural geometric action for a noncommutative geometry. The Dirac operator is the dynamical variable of the spectral action, and plays the role of the metric. A path integral should therefore be some sort of “sum over Dirac operators”. We define what this might mean by appealing to the conventional path integral formalism. Our approach builds on, and complements, the work done by Rovelli [1].

The outline of this paper is as follows. We begin in Section 2 with a detailed description of our path integral formalism. Then, in Section 3, we apply it to the two-point space, and, in Section 4, to the matrix geometry $M_2(\mathbb{C})$. For these geometries, the path integrals are standard (finite dimensional) integrals, so the technical difficulties associated with functional integration are avoided. To keep the examples clear and concise, we restrict ourselves to $(\mathcal{A}, \mathcal{H}, D)$ spectral triples. That is, we ignore real structure, orientability and Poincaré duality, which do not play an essential role in the discussion. As such, our example geometries can be considered as fragments of larger noncommutative geometries that do conform to all the axioms set out in [11].

In Section 5, we make a comparison with the canonical quantisation approach taken in [1]. The results we obtain from using our approach to quantise the geometry used in [1] are given in Section 6. An idea for a path integral approach is also proposed in [1]. In Section 7, we highlight the differences between this approach and our path integrals. We follow this, in Section 8, with a brief discussion on the quantisation of a Riemannian manifold. Section 9 marks the end, with the conclusion.

Note: we work with an Euclidean signature, i.e. Riemannian means Riemannian not pseudo-Riemannian.

2. Path integral quantisation

We decided to work on a path integral approach, rather than a canonical approach, because it requires knowledge of only the fields, and not their dynamics. To be able to canonical quantise a noncommutative geometry, we would need a general procedure for finding the phase space, and constructing a symplectic structure on it. Conventionally, this amounts to finding the canonical momenta and using the Poisson bracket. In contrast, path integrals need a (gauge invariant) measure on the space of histories. Deciding how to parameterise this space is thus an important consideration. The advantage lies in that this does not depend on the details of the action, unlike finding the phase space. The only things that really matter

are the fields, because they determine the measure. One of the other benefits of using path integrals is they are explicitly covariant.

A good starting point for developing a path integral formalism for noncommutative geometry is the conventional formalism. It has lead to standard model predictions that agree spectacularly with experiment, so it should be incorporated as a special case. Since the standard model action can be expressed in the form of a spectral action, a dictionary can be set up between noncommutative geometry and quantum field theory. This makes it apparent that the fields parameterise the Dirac operator. So, the space of histories of the fields is equivalent to the space of histories of the Dirac operator. From the noncommutative geometry point of view then, the degrees of freedom of the Dirac operator correspond to the fields in the spectral action, and hence give the path integration measure. Thus, in principle, we can path integral quantise a general spectral action. Schematically, the general partition function can be written as

$$Z = \int \mathcal{D}D e^{-\text{Tr} f(D^2/\Lambda^2)}, \tag{1}$$

where D is the Dirac operator. The function f and parameter Λ are the cutoffs for the spectral action.

3. Two-point space and Higgs gravity

The two-point space is the simplest example of a noncommutative geometry. It consists of just two points, which we label L and R. The spectral triple is given by

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C} = \left\{ f = \begin{pmatrix} f_L & 0 \\ 0 & f_R \end{pmatrix} \right\}, \quad \mathcal{H} = \mathbb{C} \oplus \mathbb{C}, \quad D = \frac{1}{\hbar} \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}, \tag{2}$$

where m is a complex constant, which fixes the distance between the two points.

Some may be unsettled by the appearance of \hbar in the Dirac operator before quantisation. It is used only to follow the convention that m has units of mass, rather than inverse length, and so can be omitted. Alternatively, one could view \hbar as the noncommutative geometry version of c . In the same way that c relates space and time on a Lorentzian manifold, \hbar relates space and (inverse) mass on a noncommutative geometry (“*spacemass*”). No \hbar is required for quantisation as the spectral action is naturally dimensionless. We, however, will take our actions to have the usual dimensions of \hbar .

To move from a static (flat) space to a dynamic (curved) space, we promote the constant m to a variable ϕ , which will play the role of the gravitational field. This is the analogue of moving from $\eta_{\mu\nu}$ to $g_{\mu\nu}(x)$ on a Lorentzian manifold. In fact, ϕ is really a connection, so it plays the role of a vierbein/spin connection rather than a metric. In the context of the standard model, ϕ is interpreted as the Higgs field, hence we refer to this as Higgs gravity.

The spectral action is taken to be

$$S = \frac{1}{G} \text{Tr} D^2 = \frac{2l_p^2}{\hbar} |\phi|^2, \tag{3}$$

where G is the gravitational coupling constant, and $l_p = 1/\sqrt{\hbar G}$ the Planck length. It has a $U(1)$ symmetry which comes from $\text{Inn}(\mathcal{A})$, the inner automorphism group of \mathcal{A} . For the two-point space, $\text{Inn}(\mathcal{A}) \cong U(1) \times U(1)$, which acts on ϕ via the $U(1)$ transformations given by the homomorphism $U(1) \times U(1) \rightarrow U(1) : (g, h) \rightarrow gh^{-1}$. The inner automorphisms are analogous to the diffeomorphisms of general relativity. They are often referred to as internal diffeomorphisms.

Varying the action, the equations of motion are simply

$$\phi = 0, \quad \bar{\phi} = 0. \tag{4}$$

Using Connes’ distance formula,

$$d(x, y) = \sup_{f \in \mathcal{A}} \{ |\langle x|f|x\rangle - \langle y|f|y\rangle| : \|[D, f]\| \leq 1 \}, \tag{5}$$

the distance between the two points is

$$d(L, R) = \sup_{f \in \mathcal{A}} \left\{ |f_L - f_R| : \frac{|\phi|^2}{\hbar^2} |f_L - f_R|^2 \leq 1 \right\} = \frac{\hbar}{|\phi|} = \frac{m_p}{|\phi|} l_p, \tag{6}$$

where m_p is the Planck mass. So, classically, the metric structure D vanishes and the distance is infinite.

Now, we quantise by doing path integrals over ϕ and $\bar{\phi}$, the degrees of freedom of D . The partition function is thus

$$Z = \int dD e^{-S'} = \int d\bar{\phi} d\phi \exp\left(-\frac{2|\phi|^2}{m_p^2}\right). \tag{7}$$

Since the action has a $U(1)$ symmetry, we shall employ some gauge-fixing. This involves nothing more than switching to polar coordinates (r, θ) , and dropping the irrelevant θ integration. Note: as the number of gauge degrees of freedom is finite, gauge-fixing is not strictly necessary (the θ integration does not give an infinite contribution). After integrating out gauge equivalent Dirac operators, the partition function reduces to

$$Z = \int_0^\infty d\phi \phi \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \frac{m_p^2}{4}, \tag{8}$$

where ϕ is now used to denote the positive real field $|\phi|$.

Expectation values are calculated in the usual fashion. For example,

$$\langle \phi \rangle = \frac{1}{Z} \int_0^\infty d\phi \phi^2 \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \frac{\sqrt{2\pi}}{4} m_p, \tag{9}$$

$$\langle d(L, R) \rangle = \frac{1}{Z} \int_0^\infty d\phi m_p l_p \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \sqrt{2\pi} l_p. \tag{10}$$

Here, we see that in the vacuum state, ϕ has acquired a v.e.v., and the distance has become finite. Though, the classical distance relation (6) no longer holds.

In general,

$$\int_0^\infty d\phi \phi^n \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{m_p}{\sqrt{2}}\right)^{n+1}. \tag{11}$$

Thus, the Greens functions are

$$\langle \phi^n \rangle = \Gamma\left(\frac{n+2}{2}\right) \left(\frac{m_p}{\sqrt{2}}\right)^n. \tag{12}$$

In particular, the propagator functions can be expressed as

$$\langle (\phi \phi)^n \rangle = n! \left(\frac{m_p^2}{2}\right)^n \tag{13}$$

for $n \in \mathbb{Z}$. These reproduce the usual propagator combinatorics (i.e. Wick contractions) for a complex scalar field.

In an excited state, the distance $d(L, R)$ is given by its expectation value in a background of propagators. So, for the N th particle state,

$$\langle d(L, R) \rangle_N = \frac{1}{Z_N} \langle \phi^N d(L, R) \phi^N \rangle, \tag{14}$$

where $Z_N = \langle (\phi \phi)^N \rangle$. This evaluates to

$$\langle d(L, R) \rangle_N = \frac{\Gamma(N + 1/2)}{\Gamma(N + 1)} \sqrt{2} l_p. \tag{15}$$

The distance thus gets successively smaller as the number of gravitons (Higgs particles) is increased. Using Stirling’s formula, we find that the distance shrinks to zero in the $N \rightarrow \infty$ limit, and so the two points merge into one. The metric D correspondingly becomes infinite, since the description of the geometry as two points is no longer valid. This resembles the behaviour of a high curvature limit, i.e. gravitational collapse to a black hole.

The spectral action can be supplemented with the fermionic term

$$S_F = \langle \bar{\psi}, D\psi \rangle = \bar{\psi}_L \phi \psi_R + \bar{\psi}_R \bar{\phi} \psi_L, \tag{16}$$

which is invariant under the full $U(1) \times U(1)$ symmetry. Note that this is purely an interaction term—the fermions are fixed at the points and do not propagate. Quantising as before, we write down the partition function,

$$Z = \int d\bar{\phi} d\phi d\bar{\psi} d\psi \exp\left(-\frac{2|\phi|^2}{m_p^2} - \langle \bar{\psi}, D\psi \rangle\right). \tag{17}$$

Remember that the Hilbert space is complex, and not Grassmann, so

$$Z = \int d\bar{\phi} d\phi \frac{1}{\det D} \exp\left(-\frac{2|\phi|^2}{m_p^2}\right) = -\int_0^\infty d\phi \frac{1}{\phi} \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \infty. \tag{18}$$

This makes the v.e.v. $\langle d(L, R) \rangle$ ill-defined, while both $\langle \phi \rangle$ and the propagator $\langle \phi \phi \rangle$ will be zero. For the excited states ($N \geq 1$), the expectation values continue to be well-behaved.

The effect of the fermions is to shield out the gravitational field, by lowering the states by one. If we were to take the tensor product of the Hilbert space with a spinor Hilbert space $L^2(\text{spin}(M))$, then the fermions would enhance the gravitational field, by raising the states.

Note: for a generic finite noncommutative geometry, the fermion contribution will be $(\det D)^{-k}$, where k is the number of fermion generations fixed by the Hilbert space.

4. Matrix geometries and gauge gravity

Next, we look at the quantisation of the simplest matrix geometry, $M_2(\mathbb{C})$. Its spectral triple is

$$\mathcal{A} = M_2(\mathbb{C}) = \left\{ f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \right\}, \quad \mathcal{H} = M_2(\mathbb{C}), \quad D = \frac{1}{\hbar} \begin{pmatrix} A_1 & A_2 \\ \bar{A}_2 & -A_1 \end{pmatrix}, \tag{19}$$

where D is an $SU(2)$ gauge field, with A_1 real and A_2 complex. The C^* -algebra can be understood as being that of the fuzzy sphere $S^2_{(n=1)}$, which only has the north and south poles as distinguishable points.

The spectral action evaluates to

$$S = \frac{1}{G} \text{Tr} D^2 = \frac{2l_p^2}{\hbar} (A_1^2 + |A_2|^2), \tag{20}$$

which is invariant under $SU(2)$ gauge transformations. Like the two-point space, the inner automorphisms $\text{Inn}(\mathcal{A}) \cong U(2)$ act on D via a homomorphism, $U(2) \rightarrow SU(2)$. The homomorphism removes the trivial $U(1)$ factor that commutes with D .

As before, we shall quantise by first gauge-fixing the action. This is most easily accomplished by changing to spherical polar coordinates. So, after dropping irrelevant factors, the partition function reads

$$Z = \int_0^\infty d\phi \phi^2 \exp\left(-\frac{2\phi^2}{m_p^2}\right) = \frac{\sqrt{2\pi}}{16} m_p^3, \tag{21}$$

where $\phi = \sqrt{A_1^2 + |A_2|^2}$. Effectively, we have chosen a gauge-fixing condition such that

$$D = \frac{1}{\hbar} \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}. \tag{22}$$

This gauge can be obtained from any other by performing an $SU(2)$ gauge transformation

$$D \rightarrow UDU^\dagger = D + U[D, U^\dagger] \tag{23}$$

with

$$U = \frac{1}{2\sqrt{\phi(\phi - A_1)}} \begin{pmatrix} \phi - A_1 + \bar{A}_2 & \phi - A_1 - A_2 \\ -(\phi - A_1 - \bar{A}_2) & \phi - A_1 + A_2 \end{pmatrix}. \tag{24}$$

The Greens functions for ϕ are

$$\langle \phi^n \rangle = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+3}{2}\right) \left(\frac{m_p}{\sqrt{2}}\right)^n. \tag{25}$$

As one would expect, they reflect the combinatorics of a field that can propagate through either a real mode ($A_1 \rightarrow A_1$) or a complex one ($A_2 \rightarrow \bar{A}_2$).

The distance between the poles of the fuzzy sphere,

$$d(1, 4) = \sup_{f \in \mathcal{A}} \{|f_1 - f_4| : \|[D, f]\| \leq 1\}, \tag{26}$$

is not as straightforward to calculate as the distance between the points of the two-point space. Evaluating the condition $\|[D, f]\| \leq 1$ gives

$$\frac{\hbar}{\phi} \geq \begin{cases} |(f_1 - f_4) + (f_2 - f_3)| \\ |(f_1 - f_4) - (f_2 - f_3)| \end{cases} \text{ depending on which is larger.} \tag{27}$$

This can be simplified by expressing it in terms of “distances” and phases,

$$\frac{\hbar}{\phi} \geq |d_{14} e^{i\alpha} \pm d_{23} e^{i\beta}|, \tag{28}$$

where $d_{14} e^{i\alpha} = (f_1 - f_4)$ and $d_{23} e^{i\beta} = (f_2 - f_3)$. Squaring up both sides, it is then easy to determine the larger lower bound,

$$\frac{\hbar^2}{\phi^2} \geq d_{14}^2 \pm 2d_{14}d_{23} \cos(\alpha - \beta) + d_{23}^2 \geq d_{14}^2 + 2d_{14}d_{23}|\cos(\alpha - \beta)| + d_{23}^2. \tag{29}$$

Hence, the upper bound on d_{14} is

$$d_{14} \leq -d_{23}|\cos(\alpha - \beta)| + \sqrt{\frac{\hbar^2}{\phi^2} - d_{23}^2 \sin^2\theta}. \tag{30}$$

Taking the supremum, the distance is therefore

$$d(1, 4) = \frac{\hbar}{\phi} = \frac{m_p}{\phi} l_p. \tag{31}$$

Similarly, we also find

$$d(2, 3) = \frac{\hbar}{\phi} = \frac{m_p}{\phi} l_p. \tag{32}$$

The expectation value of the distances, in the N th particle state, is

$$\langle d \rangle_N = \frac{\Gamma(N+1)}{\Gamma(N+3/2)} \sqrt{2} l_p. \tag{33}$$

Just like the two-point space, the distances shrink to zero in the $N \rightarrow \infty$ limit. However, the nature of this collapse is rather different. The K -groups of the fuzzy sphere do not change as it collapses to a point, indeed $K_*(M_2(\mathbb{C})) \cong K_*(\mathbb{C})$. Whereas, this is not the case for the two-point space, for which $K_*(\mathbb{C} \oplus \mathbb{C}) \cong K_*(\mathbb{C}) \oplus K_*(\mathbb{C}) \not\cong K_*(\mathbb{C})$. So, the collapse of the fuzzy sphere involves a change in commutativity, rather than topology.

From a K -theory perspective, the fuzzy sphere is more like a (noncommutative) point than a sphere. It is referred to as a sphere because of its $SU(2)$ symmetry. In fact, the space of pure states of $M_2(\mathbb{C})$ is a 2-sphere. Incidentally, the K -groups of a 2-sphere are actually isomorphic to those of the two-point space.

The fermion action for the fuzzy sphere is

$$S_F = \text{Tr} \Psi^\dagger D \Psi = \bar{\psi}_1 A_1 \psi_1 + \bar{\psi}_2 A_1 \psi_2 - \bar{\psi}_3 A_1 \psi_3 - \bar{\psi}_4 A_1 \psi_4 + \bar{\psi}_1 A_2 \psi_3 + \bar{\psi}_3 \bar{A}_2 \psi_1 + \bar{\psi}_2 A_2 \psi_4 + \bar{\psi}_4 \bar{A}_2 \psi_2. \quad (34)$$

It contains twice as many fermions as (16) due to the larger Hilbert space. The contribution to the partition function will thus be $(\det D)^{-2} = \phi^{-4}$. This will have the effect of lowering the states by two.

5. Comparison with Rovelli's canonical quantisation

We tried to compare our path integral approach with Rovelli's canonical approach (see [1] for details), but found problems with the example he gives. Rovelli modified the spectral action in an effort to obtain nontrivial equations of motion. After careful examination, we found this actually had the opposite effect. The action in question is

$$S = \frac{1}{2} \text{Tr} D \tilde{M} D = \frac{1}{2G} (\bar{m}_1 m_1 + e^{-i\theta} \bar{m}_1 m_2 + e^{i\theta} \bar{m}_2 m_1 + \bar{m}_2 m_2). \quad (35)$$

But, this can be factorised as

$$S = \frac{1}{2G} (\bar{m}_1 + e^{i\theta} \bar{m}_2)(m_1 + e^{-i\theta} m_2) = \frac{|m|^2}{2G}, \quad (36)$$

where $m = m_1 + e^{-i\theta} m_2$. We thus end up with a much simpler action and set of equations of motion. Canonical quantisation in this variable is a very different problem from the one considered by Rovelli.

Physically, the interaction terms in (35) allow the particles m_1 and m_2 to spontaneously change into one another. This is like a mixing term, so m_1 and m_2 will not make good eigenstates. As we have seen in (36), the linear combination given by m will make a good eigenstate.

Although the action (35) is not spectral per se, we can in fact still quantise it with our path integral approach. We begin by rewriting the action in terms of an effective Dirac operator, D' , so it is spectral:

$$S = \frac{1}{2} \text{Tr} D \tilde{M} D = \frac{1}{2} \text{Tr} D^\dagger P^\dagger P D = \frac{1}{2G} \text{Tr} D'^\dagger D'. \quad (37)$$

Solving $P^\dagger P = \tilde{M}$ gives

$$P = \frac{1}{\sqrt{2G}} \begin{pmatrix} 1 & e^{-i\theta} & 0 \\ 1 & e^{-i\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (38)$$

thus

$$D' = \sqrt{G}PD = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & m \\ 0 & 0 & m \\ 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

Further, a self-adjoint operator D'' can be constructed by

$$D'^{\dagger}D' = \left(\frac{D'^{\dagger} + D'}{\sqrt{2}} \right)^2 = D''^2 = \frac{1}{4} \begin{pmatrix} 0 & 0 & m \\ 0 & 0 & m \\ \bar{m} & \bar{m} & 0 \end{pmatrix}^2, \tag{40}$$

since D' is nilpotent. The degrees of freedom of D'' are m and \bar{m} , just as we have proposed. Quantising this, we end up with path integrals equivalent to those for the two-point space.

The problem with trying to canonically quantise spectral actions for finite noncommutative geometries is that they have no phase space as such. This could be taken to mean that they simply cannot be quantised, but we have shown otherwise using path integrals. Perhaps some generalisation of phase space is needed (like tangent groupoids, see [5, Section 6]), or maybe the path integral approach is just more fundamental.

We could try to reverse-engineer our path integral approach and find the canonical equivalent. The Fourier mode expansion of the Higgs graviton, in terms of creation and annihilation operators, should be

$$\hat{\phi} = \frac{m_p}{\sqrt{2}}(\hat{a} + \hat{b}^{\dagger}), \quad \hat{\phi}^{\dagger} = \frac{m_p}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{b}). \tag{41}$$

Thus, the conjugate momentum operator should be

$$\hat{\pi} = \frac{i}{\sqrt{2}m_p}(\hat{a}^{\dagger} - \hat{b}), \quad \hat{\pi}^{\dagger} = \frac{i}{\sqrt{2}m_p}(\hat{a} - \hat{b}^{\dagger}). \tag{42}$$

What, then, does the classical quantity π correspond to? We leave the further exploration of these ideas for another time.

6. Path integral quantisation of Rovelli’s geometry

Having quantised Rovelli’s modified spectral action (35) using path integrals, we shall now do the same for the un-modified spectral action

$$S = \frac{1}{G} \text{Tr}(D + JDJ^{-1})^2, \tag{43}$$

where

$$D = \frac{1}{\hbar} \begin{pmatrix} 0 & 0 & \phi_1 \\ 0 & 0 & \phi_2 \\ \bar{\phi}_1 & \bar{\phi}_2 & 0 \end{pmatrix}. \tag{44}$$

Unlike the geometries we have used in our examples, the geometry used by Rovelli does satisfy all the axioms for a real spectral triple. The eigenvalues and eigenvectors of $D + JDJ^{-1}$ are

$$\lambda = 0 : \begin{pmatrix} |\phi_2|^2 & -\phi_1\bar{\phi}_2 & 0 \\ -\bar{\phi}_1\phi_2 & |\phi_1|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{45}$$

$$\lambda = \pm \frac{2}{\hbar} \sqrt{|\phi_1|^2 + |\phi_2|^2} : \begin{pmatrix} |\phi_1|^2 & \phi_1\bar{\phi}_2 & \pm\sqrt{|\phi_1|^2 + |\phi_2|^2}\phi_1 \\ \bar{\phi}_1\phi_2 & |\phi_2|^2 & \pm\sqrt{|\phi_1|^2 + |\phi_2|^2}\phi_2 \\ \pm\sqrt{|\phi_1|^2 + |\phi_2|^2}\bar{\phi}_1 & \pm\sqrt{|\phi_1|^2 + |\phi_2|^2}\bar{\phi}_2 & |\phi_1|^2 + |\phi_2|^2 \end{pmatrix}, \tag{46}$$

so

$$S = \frac{8l_p^2}{\hbar} (|\phi_1|^2 + |\phi_2|^2). \tag{47}$$

This has a $U(2)$ symmetry under the $\text{Inn}(\mathcal{A}) \cong U(2) \times U(1)$ gauge transformations

$$D \rightarrow (UJUJ^{-1})D(UJUJ^{-1})^\dagger = D + U[D, U^\dagger] + JU[D, U^\dagger]J^{-1}. \tag{48}$$

An overall factor of $U(1)$ acts trivially because it commutes with D .

Quantising the action, we get the gauge-fixed partition function

$$Z = \int_0^\infty d\phi \phi^3 \exp\left(-\frac{8\phi^2}{m_p^2}\right) = \frac{m_p^4}{128}, \tag{49}$$

where $\phi = \sqrt{|\phi_1|^2 + |\phi_2|^2}$. From this, we find the Greens functions to be

$$\langle \phi^n \rangle = \Gamma\left(\frac{n+4}{2}\right) \left(\frac{m_p}{\sqrt{8}}\right)^n. \tag{50}$$

For the distance used in [1, Eq. (22)],

$$d = \frac{\hbar}{\sqrt{|\phi_1|^2 + |\phi_2|^2}} = \frac{m_p}{\phi} l_p, \tag{51}$$

the expectation values are

$$\langle d \rangle_N = \frac{\Gamma(N+3/2)}{\Gamma(N+2)} \sqrt{8} l_p. \tag{52}$$

7. Spectral integrals

A proposal for a path integral approach is also put forward in [1]. It suggests that the integration measure should be given by the eigenvalues of the Dirac operator. This complements

the spectral invariance of the spectral action. We shall refer to such path integrals as *spectral integrals*.

Spectral integrals differ from our path integrals in the way they integrate over the space of Dirac operators. The starting point for both is the space of self-adjoint operators, which can be partitioned into unitary equivalence classes. In our approach, we quotient out all those operators that have a nonzero trace, to leave only traceless self-adjoint operators. We then remove any degrees of freedom belonging to the centre of the C^* -algebra \mathcal{A} . This has the effect of reducing the unitary equivalence classes down to $\text{Inn}(\mathcal{A})$ equivalence classes. The space we are left with is the space of Dirac operators that we integrate over. We use gauge-fixing to perform the integration, so path integrals separate into a contribution from the gauge orbits and an integral along a section.

In contrast, spectral integrals just integrate over the orbit space of the unitary group action on the space of self-adjoint operators. The orbit space has the operator eigenvalues as Cartesian coordinates, so there is no dependence on $\text{Inn}(\mathcal{A})$. (To be precise, one should order the eigenvalues, $\lambda_1 < \lambda_2 < \dots < \lambda_n$, or include a symmetry factor in the integrals.) This means that different finite geometries with representations of the same dimension will have the same spectral integrals.

As a case in point, take the two-point space and the matrix geometry $M_2(\mathbb{C})$. Both have two-dimensional representations and so two Dirac operator eigenvalues. Their spectral integrals will therefore be identical, making it impossible to distinguish between the two geometries using expectation values alone. For example, both geometries have the distance v.e.v.

$$\langle d \rangle = \frac{1}{Z} \int d\lambda_1 d\lambda_2 \frac{\sqrt{2}l_p^2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \exp\left(-\frac{\lambda_1^2 + \lambda_2^2}{l_p^2}\right) = \sqrt{2\pi}l_p. \tag{53}$$

It should be remembered that spectral integrals are, so far, just an idea, and we have interpreted it literally. An obvious refinement that could be made is to impose a traceless condition on the eigenvalues. We await to see if there are any further developments.

8. Riemannian manifolds

We now move on to outline how our approach might work for less trivial noncommutative geometries, in particular Riemannian manifolds. The Dirac operator for a Riemannian manifold is

$$D = \gamma^a e_a^\mu(x) \left(\frac{\partial}{\partial x^\mu} + \frac{1}{4} \omega_{bc\mu}(x) \gamma^b \gamma^c \right), \tag{54}$$

where e_a^μ is the vierbein and ω_μ^{ab} is the spin connection. (Note: $\text{Tr}D = 0$, as each term contains an odd number of gamma matrices.) Computing the spectral action yields the Einstein–Hilbert action (ignoring higher order terms). The details of the calculation can be found in [2].

Usually, the metric, $g_{\mu\nu}$, is considered as the dynamical field and hence gives the measure for path integrals. In our approach, the vierbein and spin connection would be used instead,

these being the degrees of freedom of the Dirac operator. This resembles the conventional connection-based way of quantising Yang–Mills theories. So, one might hope that this will make things more tractable.

We can go further. Let us now use a Dirac operator with a self-dual spin connection A_μ^{ab} . Since we work in an Euclidean signature, A_μ^{ab} is real as $A_\mu^{ab} = 1/2\epsilon_{cd}^{ab}A_\mu^{cd}$ (it is complex in a Lorentzian signature). Applying this constraint to the spectral action will give the Einstein–Hilbert action with a self-dual curvature. This is essentially the Ashtekar formulation of general relativity.

The canonical quantisation, with respect to the spin connection, proceeds by performing a 3 + 1 ADM decomposition. From this, the conjugate momentum, π_{ab}^μ , can be determined. It is self-dual and related to the vierbein. Making use of the self-duality, one can define the variables

$$A_\mu^i = A_\mu^{0i}, \quad \pi_i^\mu = \pi_{0i}^\mu, \quad (55)$$

where $i = 1, 2, 3$ is a space index. Their Poisson bracket is

$$\{A_\mu^i(x), \pi_j^\nu(y)\} = \delta_\mu^\nu \delta_j^i \delta^3(x - y). \quad (56)$$

This is very much like the Yang–Mills situation, with i and j as the (SO(3)) group indices. There are also constraint equations, the most notorious of which, is the Hamiltonian constraint. The quantisation of the constraints is dealt with by using loop representations [12]. This is the origin of loop quantum gravity.

The path integral quantisation is related to spin foams. It is possible to write the Einstein–Hilbert action in the form of a BF theory,

$$S = \int_M e_a^\mu e_b^\nu F_{\mu\nu}^{ab} e \, d^4x = \int_M B_{ab}^{\mu\nu} F_{\mu\nu}^{ab} \sqrt{g} \, d^4x, \quad (57)$$

where $F_{\mu\nu}^{ab}$ is the self-dual curvature, and $B_{ab}^{\mu\nu} = e_a^\mu e_b^\nu$ is a constraint. Path integrals over the spin connection and vierbein then resemble the quantisation of BF theory. To make the path integrals well-defined, they can be discretised by triangulating the manifold. In BF theory, this gives rise to the concept of spin foams [13], the spin network equivalent of Feynman diagrams.

9. Conclusion

We have developed a path integral approach to quantise the spectral action. It has been successfully applied to the two-point space and the matrix geometry $M_2(\mathbb{C})$. In both cases, graviton excitations have the effect of shrinking distances. However, the behaviour of the geometries as they collapse to a point is quite different. The two-point space undergoes a topological change, which is suggestive of the formation of something like a black hole (an apt term would be “black point”). Whereas, the matrix geometry maintains its topology, but loses its noncommutativity instead. We expect the shrinking of distances by gravitons to be a general feature of quantised finite noncommutative geometries. The introduction of fermions on to the geometries has the effect of shielding out the gravitational field. All the graviton states are lowered by an amount equal to the number of fermion generations.

Comparing our approach with [1], led us to question the validity of their results. We found that their equations of motion could be expressed in much simpler terms, which result in a smaller phase space. This will alter their canonical quantisation. Despite this, both approaches seem to support the qualitative result that distances shrink with increasing graviton excitations.

The idea of spectral integrals is very appealing as it is consistent with the philosophy of spectral invariance. But, we have concerns over the possible lack of any topological dependence. The K -groups should somehow determine the (sub)space of eigenvalues to integrate over. On Riemannian manifolds, our path integral approach coincides with the conventional one, by construction. It would be interesting to see how spectral integrals differ from this.

The next step, to obtaining a better understanding of quantised noncommutative geometries, would be to investigate some more substantial examples than the ones we have considered here. For example, the spectral triple associated with the finite part of the standard model C^* -algebra, i.e. $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. We hope to pursue these ideas in the future.

Noncommutative geometry has introduced a new twist in the search for a theory of quantum gravity. The biggest problem we face may not be one of quantisation, but one of finding the right geometry to quantise.

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